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Vortex method with meshless spatial adaption for accurate simulation of viscous, unsteady vortical flows

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SUMMARY

A vortex method has been developed where spatial adaption of the Lagrangian vortex particles is provided by the technique of radial basis function interpolation. In this way, the meshless formulation of the vortex method is preserved throughout. Viscous effects are provided by the core spreading method, where core size control is accomplished in the spatial adaption, thus ensuring convergence. Numerical experiments demonstrate considerable increase in accuracy, in comparison with standard remeshing schemes used with vortex methods. Proof-of-concept is achieved successfully on a problem of quasi-steady tripole vortex flow, and parallel implementation of the method has permitted high-accuracy computations of vortex interactions at high Reynolds number. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: vortex method; meshless CFD; core spreading; viscous vortex interactions

1. INTRODUCTION

In vortex methods, the Navier–Stokes equations are expressed in vorticity formulation and the vorticity field becomes the principal variable for computations. The fluid velocity is obtained from an integral on the vorticity, and the pressure is absent from the formulation. In addition, vortex methods are characterized by using a Lagrangian approach, where vortex particles convect with the local fluid velocity. Thus, they are not formulated to depend on a computational grid, and here lies one of their potential advantages, since grid-generation can be a very expensive process of CFD (this is the case for realistic configurations and, in particular, for viscous dominated flows; see, for example Reference [1]). In addition, the Lagrangian formulation is devoid of numerically diffusive truncation errors, as shown in Reference [2], and of CFL-type stability constraints (see, for example Reference [3]). Significant reviews

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on vortex methods have been published over the years, for example References [2, 4, 5], and recently also a book [6].

1.1. Need for spatial adaption

The calculation of unsteady flows using vortex blob methods suffers from loss of accuracy due to the Lagrangian deformation of the particle field; this was shown by extensive numerical experiments in Reference [7]. As vortex particles are allowed to freely convect, 'gaps' appear in areas where the flow strain is large and thus the particle set loses the ability to reconstruct accurately the smooth vorticity field. The widespread solution to this issue is the application of a 'remeshing' or 'redistribution' scheme, whereby particles are periodically re-located onto a regular lattice, and their circulation weights are interpolated to the new locations. The addition of remeshing to Lagrangian vortex methods made long-time, unsteady calculations possible [8], and has been embraced by various workers who have produced remarkable results [9–13]. On the other hand, some authors have been concerned that the grid-free nature of vortex methods is undermined by the requirements of remeshing. More importantly, the simulation of high-Reynolds number flows to high accuracy may be limited by the errors of interpolation in the remeshing schemes. An extensive program of numerical experiments has demonstrated that remeshing does introduce noticeable errors; see Reference [14].

1.2. Meshless alternative to standard remeshing

The remeshing processes in widespread use obtain values of circulation on the nodes of a regular stencil by interpolation of the circulation strength of each scattered particle, using tensor product formulations. Thus, they apply interpolation from a scattered set of nodes onto a regular one, with a moderate stencil width. It has been shown, however, that the most accurate techniques for the interpolation of scattered data are those with global influence. In a renowned review paper [15] studying almost 30 algorithms, the methods of interpolation using multi-quadrics and thin-plate splines were declared superior in accuracy; these are examples of (global) radial basis function (RBF) interpolation.

The techniques of RBF interpolation were recently applied by the authors to the spatial adaption of Lagrangian vortex particles in a time-marching algorithm, and the method was tested and compared to the standard remeshing schemes, using classic test problems; see Reference [14]. The potential of considerable increase in accuracy, with a fully meshless method, was demonstrated in said tests. In one of the reported experiments, RBF interpolation resulted in a four-order-of-magnitude improvement over remeshing using a standard third-order kernel. In addition to this, the use of RBF interpolation for spatial adaption provided an opportunity to use the core spreading viscous scheme [2, 16], with core size control enforced during the adaption process. In this way, the convergence issues of core spreading pointed out in Reference [17] are successfully addressed. Until now, the only available solution to said convergence issue was vortex particle splitting [18]; see below.

1.3. Viscous schemes in vortex methods

Historically, the first vortex method to include viscous effects was the random vortex method (RVM) [19], where in a fractional step formulation, the viscous sub-step consists of a Brownian-like motion of the particles. The slow rate of convergence of this stochastic method motivated the development of various alternatives. Possibly the prevailing 'deterministic' viscosity scheme is the particle strength exchange method (PSE), where, as in general particle methods [20], the Laplacian at a particle's location is approximated by an integral operator which is discretized by quadrature, using as quadrature points the locations of the particles. In principle, the formulation of PSE is grid-free, but the fact that the accuracy relies strongly on the quadrature rules used for the discretized integral means that in practice the method hinges on having nearly uniformly spaced particle locations. For this reason, the extensive implementation of the PSE method has promoted the development and widespread use of remeshing schemes, as discussed above. This has caused a bit of debate, as some workers maintain that the grid-free nature of the vortex method is undermined when remeshing schemes relying on regular particle grids are applied, sometimes as often as every time step.

Contemporary to the introduction of the RVM, it was recognized that vortex blobs could be allowed to grow in time to simulate viscosity [2, 16], which came to be known as the core spreading method. In this method, one can view the discretized vorticity field as a superposition of 'spreading line vortices' of different circulation strengths. Hence, to satisfy identically the viscous part of the vorticity equation one lets the blob cores grow in time. This method is utterly simple to implement, it is fully localized and grid-free, and easily parallelizable—as is the RVM, but core spreading is deterministic so in principle allows for better error control and faster convergence than the RVM. However, research and use of the core spreading method stalled when the method was declared inconsistent in Reference [17]. The consistency error of core spreading is caused by the advection without deformation of larger and larger vortex blobs as they spread. A particle splitting scheme was finally proposed to alleviate this problem in Reference [18], and proved to be convergent. Research on the core spreading vortex method was effectively 'resurrected', but the splitting does introduce numerical diffusion and its errors are in fact comparable to the RVM.

In this paper, proof-of-concept calculations of the evolution of a perturbed Gaussian monopole, possessing a quasi-steady tripole attractor, are presented. This problem has been studied before using vortex methods with core spreading and splitting [21], and our results agree with these studies in general but demonstrate considerably improved quality. By means of a parallel implementation of the method, a highly accurate calculation of vortex interaction at high Reynolds number was also performed, with results that compare very well with published spectral methods calculations [22].

2. FORMULATION OF THE METHOD

For incompressible flow, $\nabla \cdot \mathbf{u}(\mathbf{x}, t) = 0$, and the two-dimensional vorticity transport equation is

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = \nu \Delta \omega \tag{1}$$

The vortex blob method proceeds by spatially discretizing the vorticity field onto particles which have a radially symmetric distribution of vorticity, thus

$$\omega(\mathbf{x}, t) \approx \omega^h(\mathbf{x}, t) = \sum_{i=1}^{N} \Gamma_i(t) \zeta_{\sigma_i}(\mathbf{x} - \mathbf{x}_i(t)) \tag{2}$$

where $\mathbf{x}_i$ is the particle position, Γ_i is its circulation strength, and the core size is σ_i. The core sizes are usually uniform ($\sigma_i = \sigma$), and the characteristic distribution of vorticity ζ_{σ_i} commonly called the cut-off function, is frequently a Gaussian distribution, such as

$$\zeta_\sigma(\mathbf{x}) = \frac{1}{2\pi\sigma^2} \exp\left(\frac{-|\mathbf{x}|^2}{2\sigma^2}\right) \tag{3}$$

The velocity field is obtained from the vorticity by means of the Biot–Savart law

$$\mathbf{u}(\mathbf{x},t) = \int (\nabla \times \mathbf{G})(\mathbf{x}-\mathbf{x}')\omega(\mathbf{x}',t)\,\mathrm{d}\mathbf{x}' = \int \mathbf{K}(\mathbf{x}-\mathbf{x}')\omega(\mathbf{x}',t)\,\mathrm{d}\mathbf{x}' = (\mathbf{K} * \omega)(\mathbf{x},t)$$

where $\mathbf{K} = \nabla \times \mathbf{G}$ is the Biot–Savart kernel, with $\mathbf{G}$ the Green's function for the Poisson equation, and $*$ representing convolution.

The vorticity transport is solved in this discretized form by convecting the particles with the local fluid velocity, and accounting for viscous effects by changing the particle vorticity. Hence, the unbounded vortex method is expressed by the following system of equations:

$$\frac{\mathrm{d}\mathbf{x}_i}{\mathrm{d}t} = \mathbf{u}(\mathbf{x}_i,t) = (\mathbf{K} * \omega)(\mathbf{x}_i,t), \qquad \frac{\mathrm{d}\omega}{\mathrm{d}t} = v\nabla^2\omega \tag{4}$$

To provide viscous effects via the core spreading method, the discretized vorticity field is viewed as a superposition of spreading line vortices (see Reference [23, p. 204])

$$\omega_i(\mathbf{x}_i,t) = \frac{\Gamma_i}{4\pi vt} \exp\left(\frac{-|\mathbf{x}_i|^2}{4vt}\right) \tag{5}$$

Comparing (5) with (3) it is seen that the linear diffusion equation can be satisfied exactly by spreading σ^2 according to

$$\frac{\mathrm{d}\sigma^2}{\mathrm{d}t} = 2v \tag{6}$$

which means that each vortex blob must spread out at a rate proportional to $\sqrt{vt}$, or $\sqrt{v\Delta t}$ at each time step. Note that this method does not apply viscous splitting of the Navier–Stokes equation.

3. MESHLESS SPATIAL ADAPTION

A spatial adaption algorithm for the vortex method has been developed, based on radial basis function (RBF) interpolation [24]. This is a technique for scattered data interpolation that has received abundant interest in the function approximation community in the past years. The scheme is formulated wholly grid-free, and numerical experiments demonstrate that it provides opportunities for increased accuracy in vortex methods, compared to standard remeshing schemes. In Reference [14], it was shown how with a very accurate discretization of a radially symmetric vorticity distribution consisting of a Lamb–Oseen vortex, it can be seen clearly that the first processing with remeshing causes a jump in the errors. These tests were performed using the so-called M_4' interpolation kernel, a third-order scheme first introduced for SPH methods [25], which is currently the preferred method of many workers. The spatial adaption with RBF interpolation does not exhibit this 'initial remesh error' and was able to provide up to a four-order-of-magnitude improvement in velocity errors in one of the tests presented in Reference [14].

The problem of scattered data interpolation is that of how to approximate an unknown function $f \in C(\Omega)$ whose values are known on a set $X = \{x_1, \ldots, x_N\} \subset \Omega \subset \mathbb{R}^d$. The RBF approach, following the notation of Reference [26], is to choose the function that approximates f to be of the form

$$s_{f,X}(x) = \sum_{j=1}^{N} \alpha_j \ \Phi_j(x, x_j) + p(x) \tag{7}$$

where $p(x)$ is a low-degree polynomial, and $\Phi : \Omega \times \Omega \to \mathbb{R}$ is a fixed function satisfying translation invariance and radiality. Clearly, the blob discretization of the vorticity is analogous to interpolant (7), where the polynomial part is chosen as null and the basis function is the cut-off function.

The solution of (7) requires the satisfaction of the interpolation conditions by collocation, leading to a linear system $\mathbf{\Phi}\alpha = \vec{f}$ for the coefficients $\alpha = (\alpha_1, \ldots, \alpha_N)$, where $\vec{f}$ is the vector of function values at the centres, $\vec{f} = \{f(x_1), \ldots, f(x_N)\}$, and $\mathbf{\Phi}_{ij} = \phi(\|x_i - x_j\|)$. The matrix $\mathbf{\Phi}$ being full and ill-conditioned, it was concluded in Reference [15] that global basis function methods are not feasible for large N. Significant progress since then includes the use of multipole expansions for the fast evaluation of interpolant (7), and for the matrix-vector product on each step of an iterative solution method [27], and the use of preconditioning and a GMRES iterative solution method [28], among others. Thus, large problem sizes have become tractable.

During a spatial adaption process, a new set of particles is laid out inside the bounding box of the current particle set. For simplicity (since our applications are unbounded), the new particles are arranged in regular triangular (or sometimes square) lattices; however, by formulation, a regular lattice is not required. The old particle information is used to calculate the vorticity on the location of each new particle; this constitutes the left-hand side of system (7). Then, the circulation strengths of the new particles are found by solving the linear system, where the matrix $\mathbf{\Phi}_{ij} = \zeta_\sigma(x_i - x_j)$. In constructing this matrix, the core sizes are set to the original value before the core spreading steps (thus, the frequency of spatial adaption determines the maximum core size in the calculation).

4. CALCULATIONS OF VISCOUS VORTEX INTERACTIONS

Extensive numerical studies have been performed where, using classic axisymmetric test problems such as inviscid vortex patches and Lamb–Oseen vortices, the proposed method has been compared with the standard vortex method using remeshing. First results were presented in Reference [14], and further detailed results including convergence studies will be presented elsewhere; they must be omitted here due to space limits, and the preference for including results of more fluid dynamical interest apropos of viscous vortex interactions. Our first result is the calculation of the flow resulting from adding a localized elliptic perturbation to a Gaussian vortex. When the perturbation amplitude is large, in Reference [21] it was demonstrated that

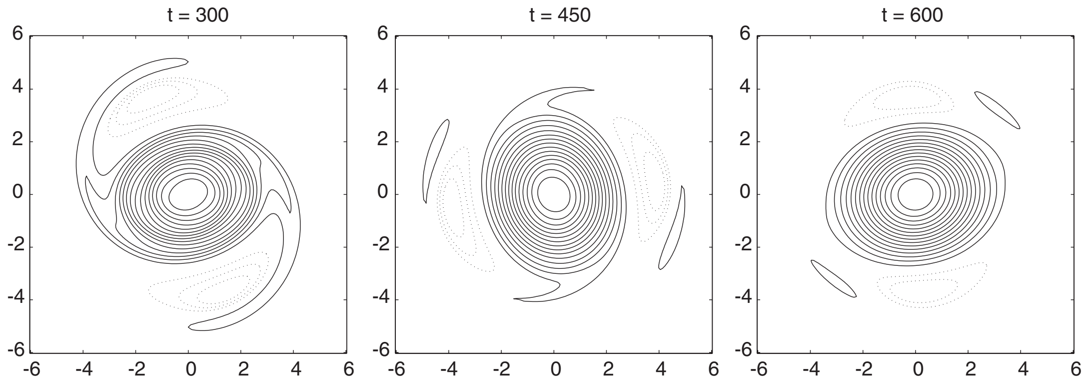

Figure 1. Perturbed monopole relaxing to a tripole. $Re = 10^4$, strength of the perturbation $\delta = 0.25$; 15 equally spaced positive contours, and three equally spaced negative contours in dotted line.

this flow relaxes to a quasi-steady rotating tripole. The initial condition is an axisymmetric vortex ω_0 with perturbation ω', given below (where $\theta = \arg \mathbf{x}$ is the azimuthal angle):

$$\omega_0(\mathbf{x}) = \frac{1}{4\pi} \exp\left(\frac{-|\mathbf{x}|^2}{4}\right), \quad \omega'(\mathbf{x}) = \frac{\delta}{4\pi} |\mathbf{x}|^2 \exp\left(\frac{-|\mathbf{x}|^2}{4}\right) \cos 2\theta \tag{8}$$

The relaxation to a quasi-steady tripole is shown in Figure 1. These results reproduce well those in Reference [21] in terms of shape of contours and rotation angle, but are much smoother. We also obtain a tripole for the smaller $Re = 10^3$, for which in Reference [21] the structure erodes due to diffusion: we argue that this is numerical and caused by the errors and numerical diffusion due to their vortex splitting scheme.

The calculation shown in Figure 1 was performed with: initial inter-particle spacing, $h = 0.18$; initial core size, $\sigma = 0.2$; number of particles $N < 5000$. Spatial adaption was performed every 10 time steps, so that $\sigma_{\max} = 0.2049$, and time stepping was provided by a fourth-order Runge–Kutta scheme, with $\Delta t = 1.0$. Unfortunately, there are no numerical parameters reported in Reference [21], but the first author informally stated to us that the number of vortex elements in their computations were in the 'low to middle 10's of thousands' (L. F. Rossi, private communication, 2004). One can safely assume that they used more than double the number of particles than in the calculation shown in Figure 1; their results, however, are noticeably noisy in comparison.

A direct comparison was made of the accuracy of (third-order) remeshing vs RBF interpolation on this flow. Looking at the field difference of vorticity on a sampling mesh, before and after spatial adaption, one can obtain an error measure. Using in this case $h = 0.2$, and shutting off core spreading (i.e. inviscid), the error for three different remeshing events was of order 10^{-6}; using RBF interpolation instead, the error was of order 10^{-10}. Finally, a grid-refinement study was performed using $h_1 = 0.09$, $h_2 = 0.126$, $h_3 = 0.1764$ (i.e. grid-refinement ratio $r = 1.4$), and an observed order of convergence was obtained of $p = 2$, which is consistent with the estimates of convection error as developed in Reference [2] (more details are given in Reference [29]).

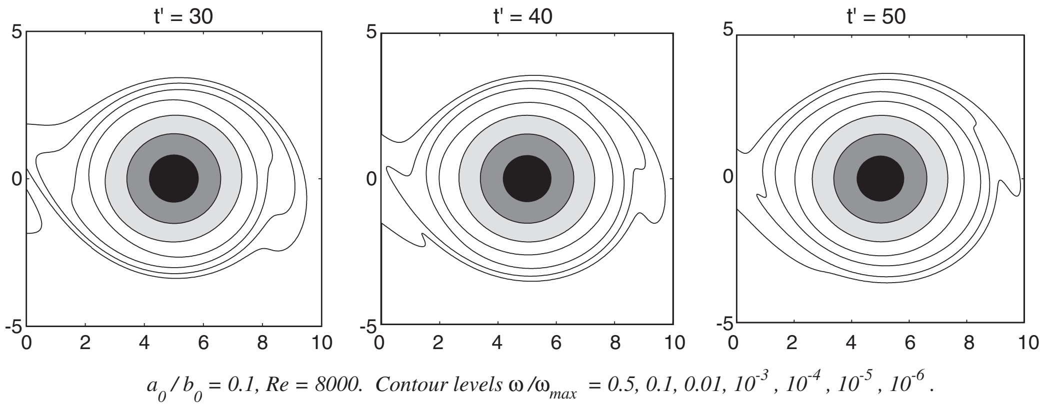

Figure 2. Vorticity contours of the right side of a symmetric two-vortex system (initially Gaussian) as they adapt to each other's strain field (on a frame rotating with the vortices); $t' = t\Gamma/(2\pi a_0^2)$.

The second application was obtained with a parallel implementation of the method, using the PETSc library [30] and its built-in GMRES solvers. The flow corresponds to two Gaussian vortices, with ratio between their core size and separation distance, $a_0/b_0 = 0.1$, and Figure 2 shows the core deformations due to their early interaction. This reproduces extremely well the results of Reference [22] using spectral methods on a 1024^2 mesh; the number of particles in this calculation was only 2×10^4. Note the logarithmic contour levels: this is a very severe test for any numerical method, and thus the capacity for high accuracy with a meshless method of computation is demonstrated.

5. CONCLUSION

A viscous vortex method has been developed where spatial adaption of the Lagrangian particles is effected by means of radial basis function interpolation, resulting in significantly increased accuracy in comparison with standard vortex methods with remeshing. The method has been implemented in parallel, and validation is provided using problems of viscous vortex interactions at high Reynolds numbers, including the evolution of a quasi-steady rotating tripole, and the early interaction of two co-rotating viscous vortices—a severe test, for which published results using spectral methods have been reproduced, demonstrating the high accuracy of the new method. The formulation is wholly meshless, furthermore, and potentially allows multi-scale computations and variable resolution in the physical domain. These extensions are elements of the research in progress.

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